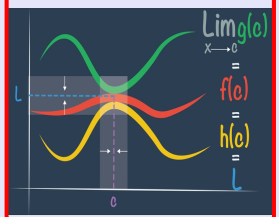
Calculus I

Lecture 19



Feb 19-8:47 AM

Class QZ 14

Given $f(x) = x(x-4)^3$, $f'(x) = 4(x-4)^2(x-1)$, $f''(x) = 12(x-4)(x-2)$

1) Find all intercepts $(0,0)$, $(4,0)$

2) Find all critical numbers ($f'(x) = 0$ or undefined)

$x-4=0$ or $x-1=0$
$x=4$ $\quad$ $x=1$

3) Find all possible inflection Points ($f''(x)=0$ or undef.)

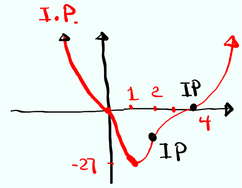
$x-4=0$ or $x-2=0$
$x=4$ $\quad$ $x=2$

4) Make a Complete Sign chart for $f'(x)$, $f''(x)$, and $f(x)$.

x	$-\infty$		1		2		4		∞
$f'(x)$		$-$	•	$+$		$+$	•	$+$	
$f''(x)$		$+$		$+$	•	$-$	•	$+$	
$f(x)$		↘		↗		↗		↗	
					I.P.		I.P.		

$f(x) = x(x-4)^3$

$(1,-27)$ $\quad$ $(2,-16)$ $\quad$ $(4,0)$



Jul 16-7:28 AM

Given: $f(x)=\frac{x}{\sqrt{x^2+1}}$, $f'(x)=\frac{1}{(x^2+1)^{3/2}}$, $f''(x)=\frac{-3x}{(x^2+1)^{5/2}}$

1) Domain $(-\infty,\infty)$

$x^2+1>0$

2) V.A. None

3) All intercepts $(0,0)$

4) $\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}\ \frac{\infty}{\infty}$

$=\lim_{x\to\infty}\frac{\frac{x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}=1$ H.A. $y=1$

$x\to\infty \Rightarrow x=\sqrt{x^2}$

$x\to-\infty \Rightarrow x=-\sqrt{x^2}$

5) $\lim_{x\to-\infty} f(x)=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}=\frac{\infty}{\infty}$

$=\lim_{x\to-\infty}\frac{\frac{x}{x}}{-\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}=-1$ H.A. $y=-1$

6) Even, odd, or neither $f(-x)=\frac{-x}{\sqrt{(-x)^2+1}}=\frac{-x}{\sqrt{x^2+1}}=-f(x)$

odd, Symmetric w/t origin

7) Critical pts

$f'(x)=0$ or und. $\rightarrow$ None

8) P.I.P.

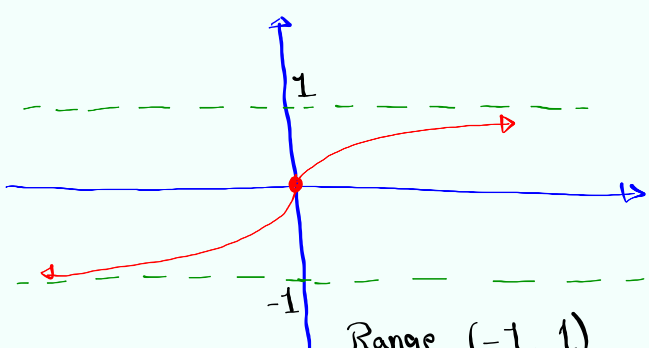
$f''(x)=0$ or und. $\rightarrow (0,0)$

$-3x=0$ $\quad x=0$

Jul 17-8:33 AM

$f'(x)=\frac{1}{(x^2+1)^{3/2}}$, $f''(x)=\frac{-3x}{(x^2+1)^{5/2}}$

x	$-\infty$	0	∞
$f'(x)$	+		+
$f''(x)$	+	•	−
$f(x)$	↗		↗



Range $(-1, 1)$

Jul 17-8:56 AM

Given $f(x)=\frac{x^3}{x-2}$, $f'(x)=\frac{2x^2(x-3)}{(x-2)^2}$, $f''(x)=\frac{2x(x^2-6x+12)}{(x-2)^3}$

1) Domain $(-\infty,2)\cup(2,\infty)$ 2) V.A. $x=2$

$x-2\neq 0$

$x\neq 2$

3) All intercepts $(0,0)$

4) $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\frac{x^3}{x-2}=\infty$

Divide by x

$\lim\limits_{x\to\infty}\frac{x^2}{1-\frac{2}{x}}=\infty$

NO H.A.

5) $\lim\limits_{x\to-\infty} f(x)=\infty$

6) C.P. $(0,0)$

$f'=0$ or und. $(3,27)$

$(2,\ \ )$ Not in the domain

7) P.I.P.

$f''=0$ or und. $(0,0)$

$2x(x^2-6x+12)=0$

$x=0$ $\quad x^2-6x+12=0$

$\underbrace{x^2-6x+9}+3=0$

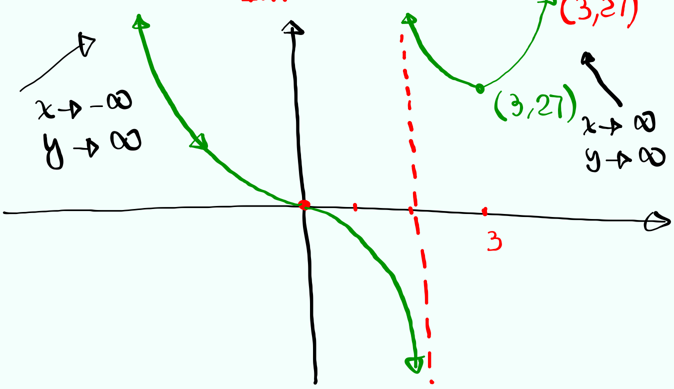
$(x-3)^2+3\neq 0$

Jul 17-9:01 AM

$f'(x)=\frac{2x^2(x-3)}{(x-2)^2}$ $\qquad f''(x)=\frac{2x\overbrace{(x^2-6x+12)}^{+}}{(x-2)^3}$ $\quad (x-3)^2+3$

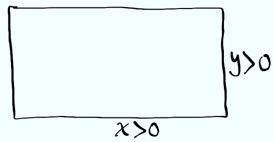
x	$-\infty$		0		2		3		∞
f'		$-$	●	$-$	O	$-$	●	$+$	
f''		$+$	●	$-$	O	$+$		$+$	
f									

I.P. $\qquad (3,27)$



Jul 17-9:23 AM

Find dimensions of a rectangle with area $1000\ m^2$ with $\underbrace{\text{Smallest Perimeter}}$.



$A = \boxed{xy = 1000}$ $\quad y = \frac{1000}{x}$

$\underbrace{P = 2x + 2y}_{\text{minimum}}$

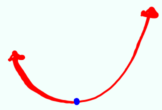
$P = 2x + 2\left(\frac{1000}{x}\right)$

$f(x) = 2x + \frac{2000}{x}$

$f'(x) = 2 - \frac{2000}{x^2}$

$f''(x) = \frac{4000}{x^3} > 0 \rightarrow CU$

Since $x > 0$



$f'(x) = 0$

$2 - \frac{2000}{x^2} = 0$

$2x^2 - 2000 = 0$

$x^2 - 1000 = 0$

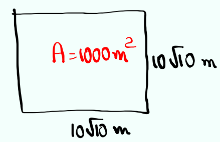
$x^2 = 1000$

$x = \sqrt{1000}$

$x = 10\sqrt{10}$

$y = \frac{1000}{x} = \frac{\overset{100}{\cancel{1000}}}{\cancel{10}\sqrt{10}}$

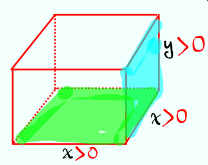
$= \frac{100}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = 10\sqrt{10}$



The rectangle with smallest perimeter is actually a square.

Jul 17-9:47 AM

A box with a square base and open top must have a volume of $32000\ cm^3$.



Volume of a box

L W H

$x\ x\ y$

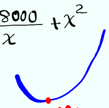
$\boxed{x^2 y = 32000}$

$y = \frac{32000}{x^2}$

Find dimensions of this box with least amount of materials to build.

$\underset{\text{Sides}}{4xy} + \underset{\text{Bottom}}{x^2} = 4x \cdot \frac{32000}{x^2} + x^2$

$f(x) = \frac{128000}{x} + x^2$



$f'(x) = \frac{-128000}{x^2} + 2x$

$f''(x) = \frac{256000}{x^3} + 2 > 0 \rightarrow CU$

for $x > 0$

$f'(x) = 0$

$\frac{-128000}{x^2} + 2x = 0$

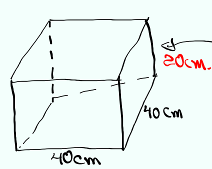
$-128000 + 2x^3 = 0$

$2x^3 = 128000$

$x^3 = 64000$

$x = \sqrt[3]{64000}$

$x = 40$



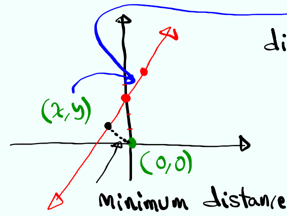
$y = \frac{32000}{x^2}$

$= \frac{32000}{40^2} = \frac{\overset{2}{\cancel{32000}}}{\cancel{1600}} = 20$

$\boxed{40 \times 40 \times 20 cm}$

Jul 17-9:56 AM

Find a point on the line $\underbrace{y=2x+3}_{\text{slope - Int}}$ that is the closest to the origin.



distance Formula between two Points

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$d=\sqrt{(x-0)^2+(y-0)^2}$

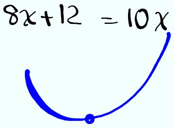
$=\sqrt{x^2+y^2}$

$=\sqrt{x^2+(2x+3)^2}$

$f(x)=x^2+(2x+3)^2$

$f'(x)=2x+2(2x+3)\cdot 2=2x+8x+12=10x+12$

$f''(x)=10>0 \rightarrow CU$



Min

$f'(x)=0$

$10x+12=0$

$10x=-12$

$\boxed{x=-1.2}$

$y=2x+3$

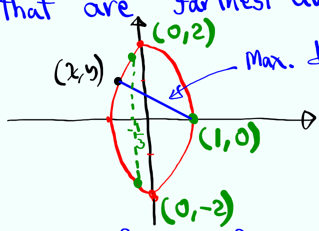
$=2(-1.2)+3$

$=-2.4+3=\boxed{.6}$

$\boxed{\text{Point } (-1.2, .6)}$

Jul 17-10:10 AM

Find point(s) on the ellipse $4x^2+y^2=4$ that are farthest away from $(1,0)$



$d=\sqrt{(x-1)^2+(y-0)^2}$

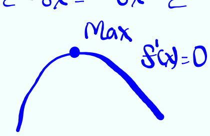
$=\sqrt{(x-1)^2+y^2}$

$=\sqrt{(x-1)^2+4-4x^2}$

$f(x)=(x-1)^2+4-4x^2$

$f'(x)=2(x-1)\cdot 1+0-8x=2x-2-8x=-6x-2$

$f''(x)=-6<0 \Rightarrow CD$



$-6x-2=0$

$\boxed{x=\frac{-1}{3}}$

$4x^2+y^2=4$

$4\left(\frac{-1}{3}\right)^2+y^2=4$

$\frac{4}{9}+y^2=4$

$y^2=4-\frac{4}{9}=\frac{32}{9}$

$y=\pm\sqrt{\frac{32}{9}}=\pm\frac{4\sqrt{2}}{3}$

Points

$\left(\frac{-1}{3}, \frac{4\sqrt{2}}{3}\right), \left(\frac{-1}{3}, \frac{-4\sqrt{2}}{3}\right)$

Jul 17-10:22 AM

MVT

1) $f(x)$ is Cont. on $[a,b]$

2) $f(x)$ is diff. on (a,b)

there exist at least one number c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

ex: $f(x)=x^3-x$, $[0,2]$

Polynomial Function $\Rightarrow$ Cont. & diff $(-\infty,\infty)$

$f'(x)=3x^2-1$

$f'(c)=3c^2-1$

$f(0)=0$

$f(2)=6$

$f'(c)=\dfrac{f(2)-f(0)}{2-0}$

$3c^2-1=\dfrac{6-0}{2-0}$

$3c^2-1=3$

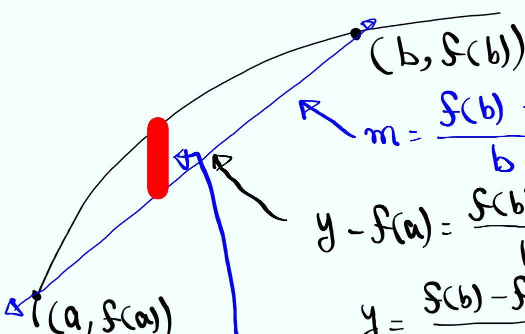
$3c^2=4$

$c^2=\dfrac{4}{3}$

$c=\pm\dfrac{2}{\sqrt{3}}$

$c\approx 1.155$ is in $(0,2)$

Jul 17-10:34 AM



$m=\dfrac{f(b)-f(a)}{b-a}$

$y-f(a)=\dfrac{f(b)-f(a)}{b-a}(x-a)$

$y=\dfrac{f(b)-f(a)}{b-a}(x-a)+f(a)$

$H(x)=y_{curve}-y_{line}$

$$\boxed{H(x)=f(x)-\frac{f(b)-f(a)}{b-a}(x-a)-f(a)}$$

1) $H(x)$ is Cont. on $[a,b]$

2) $H(x)$ is diff. on (a,b)

3) $H(a)=0$, $H(b)=0$

Conditions of Rolle's Thrm

Jul 17-10:41 AM

Conclusion of Rolle's Thrm $H'(c) = 0$

$$H(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$$

$$H'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \cdot 1 - 0$$

$$H'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \qquad H'(c) = 0$$

$$f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Jul 17-10:47 AM

Given $3 \le f'(x) \le 5$ for all values of x.

Show $18 \le f(\underset{b}{8}) - f(\underset{a}{2}) \le 30$.

Use MVT

$$f'(c) = \frac{f(8)-f(2)}{8-2}$$

$$3 \le \frac{f(8)-f(2)}{\underbrace{8-2}_{6}} \le 5$$

$$6 \cdot 3 \le f(8) - f(2) \le 6 \cdot 5$$

$$\boxed{18 \le f(8) - f(2) \le 30}$$

Jul 17-10:51 AM

Show $|\sin b - \sin a| \le |b - a|$

Hint: Use MVT, $[a,b]$, $f(x) = \sin x$

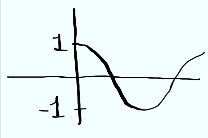
1) Is $f(x) = \sin x$ cont on $[a,b]$? ✓

2) Is $f(x) = \sin x$ diff on (a,b)? ✓

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f'(x) = \cos$

$$\cos c = \frac{f(b) - f(a)}{b - a} = \frac{\sin b - \sin a}{b - a}$$



$|\cos x| \le 1$

$$|\cos c| \le 1$$

$$\left|\frac{\sin b - \sin a}{b - a}\right| \le 1$$

$$\frac{|\sin b - \sin a|}{|b - a|} \le 1$$

We needed to show $\longrightarrow$ $\boxed{|\sin b - \sin a| \le |b - a|}$

Jul 17-10:55 AM